SUBTEMA: PROCESO DE GRAM-SCHMIDT

Problema 1: Sean $P_{\le 2}$ el espacio vectorial real de los polinomios de grado menor o igual a dos con coeficientes reales, $B=\{1,x,x^2\}$ una base de $P_{\le 2}$ y el producto interno en $P_{\le 2}$ definido por:

$$\left(p|q\right)=\int_{-1}^{1}p(x)q(x)dx$$

(a) A partir de *B*, determinar una base ortogonal de $P_{\le 2}$.

(b) Obtener el vector de coordenadas de $h(x)=1+2x-3x^2$ en la base ortogonal del inciso anterior.

SOLUCIÓN:

(a) ¿La base *B* es ortogonal?

$$\left(\overline{v_1}|\overline{v_2}\right)=\int_{-1}^{1}1(x)dx=\int_{-1}^{1}xdx=\left[\frac{x^2}{2}\right]_{-1}^{1}=\frac{1}{2}-\frac{1}{2}=\boxed{0}$$

$$\left(\overline{v_1}|\overline{v_3}\right)=\int_{-1}^{1}1(x^2)dx=\int_{-1}^{1}x^2dx=\left[\frac{x^3}{3}\right]_{-1}^{1}=\frac{1}{3}-\left(-\frac{1}{3}\right)=\frac{1}{3}+\frac{1}{3}=\boxed{\frac{2}{3}}\neq 0 \leftarrow B \text{ no es ortogonal}$$

- Mediante el proceso de Gram-Schmidt:

$$\overline{w_1}=\overline{v_1} \quad\rightarrow\quad \therefore\ \boxed{\overline{w_1}=1}$$

$$\overline{w_2}=\overline{v_2}-\frac{\left(\overline{v_2}|\overline{w_1}\right)}{\left(\overline{w_1}|\overline{w_1}\right)}\overline{w_1}$$

$$\left(\overline{v_2}|\overline{w_1}\right)=\int_{-1}^{1}x(1)dx=\int_{-1}^{1}xdx=\boxed{0}$$

$$\left(\overline{w_1}|\overline{w_1}\right)=\int_{-1}^{1}1(1)dx=\int_{-1}^{1}dx=\left[x\right]_{-1}^{1}=1-(-1)=\boxed{2}$$

$$\therefore\ \overline{w_2}=x-\frac{0}{2}(1) \quad\rightarrow\quad \boxed{\overline{w_2}=x}$$

$$\overline{w_3} = \overline{v_3} - \frac{\left(\overline{v_3}\middle|\overline{w_1}\right)}{\left(\overline{w_1}\middle|\overline{w_1}\right)}\overline{w_1} - \frac{\left(\overline{v_3}\middle|\overline{w_2}\right)}{\left(\overline{w_2}\middle|\overline{w_2}\right)}\overline{w_2}$$

$$\left(\overline{v_3}\middle|\overline{w_1}\right) = \int_{-1}^{1} 1(x^2)dx = \int_{-1}^{1} x^2 dx = \boxed{\frac{2}{3}}$$

$$\left(\overline{v_3}\middle|\overline{w_2}\right) = \int_{-1}^{1} x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = \boxed{0}$$

$$\left(\overline{w_2}\middle|\overline{w_2}\right) = \int_{-1}^{1} x(x)dx = \int_{-1}^{1} x^2 dx = \boxed{\frac{2}{3}}$$

$$\therefore\ \overline{w_3} = x^2 - \frac{2/3}{2}(1) - \frac{0}{2/3}(x) \quad \rightarrow \quad \boxed{\overline{w_3} = x^2 - \frac{1}{3}}$$

- Por tanto:

$$\boxed{B_{OG} = \left\{1, x, x^2 - \frac{1}{3}\right\}} \leftarrow \text{ Base ortogonal}$$

(b) El Vector de coordenadas en la base ortogonal B_{OG} del inciso anterior buscado es: $(h)_{BOG} = \begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix}$; donde sus coordenadas se obtienen con: $\alpha_1 = \frac{\left(h\middle|\overline{w_1}\right)}{\left(\overline{w_1}\middle|\overline{w_1}\right)}$, $\alpha_2 = \frac{\left(h\middle|\overline{w_2}\right)}{\left(\overline{w_2}\middle|\overline{w_2}\right)}$ y $\alpha_3 = \frac{\left(h\middle|\overline{w_3}\right)}{\left(\overline{w_3}\middle|\overline{w_3}\right)}$.

- Calculando los productos internos correspondientes:

$$\alpha_1 = \frac{\left(h\middle|\overline{w_1}\right)}{\left(\overline{w_1}\middle|\overline{w_1}\right)};\quad \left(h\middle|\overline{w_1}\right) = \int_{-1}^{1}(1+2x-3x^2)(1)dx = \int_{-1}^{1}(1+2x-3x^2)dx =$$

$$= \left[x + x^2 - x^3\right]_{-1}^{1} = 1+1-1-(-1+1+1) \rightarrow \boxed{\left(h\middle|\overline{w_1}\right) = 0}$$

$$\boxed{\left(\overline{w_1}\middle|\overline{w_1}\right) = 2}$$

$$\therefore\ \boxed{\alpha_1 = 0}$$

$$\alpha_2=\frac{\left(h\middle|\overline{w_2}\right)}{\left(\overline{w_2}\middle|\overline{w_2}\right)};\quad \left(h\middle|\overline{w_2}\right)=\int_{-1}^{1}(1+2x-3x^2)(x)dx=\int_{-1}^{1}(x+2x^2-3x^3)dx=$$

$$=\left[\frac{x^2}{2}+\frac{2}{3}x^3-\frac{3}{4}x^4\right]_{-1}^{1}=\frac{1}{2}+\frac{2}{3}-\frac{3}{4}-\left(\frac{1}{2}-\frac{2}{3}-\frac{3}{4}\right)\rightarrow \boxed{\left(h\middle|\overline{w_2}\right)=\frac{4}{3}}$$

$$\boxed{\left(\overline{w_2}\middle|\overline{w_2}\right)=\frac{2}{3}}$$

$$\alpha_2=\frac{\frac{4}{\not 3}}{\frac{2}{\not 3}}=\frac{4}{2}\quad\rightarrow\quad\therefore\boxed{\alpha_2=2}$$

$$\alpha_3=\frac{\left(h\middle|\overline{w_3}\right)}{\left(\overline{w_3}\middle|\overline{w_3}\right)};\quad \left(h\middle|\overline{w_3}\right)=\int_{-1}^{1}(1+2x-3x^2)\left(x^2-\frac{1}{3}\right)dx=\int_{-1}^{1}\left(x^2+2x^3-3x^4-\frac{1}{3}-\frac{2}{3}x+x^2\right)dx=$$

$$=\int_{-1}^{1}\left(-\frac{1}{3}-\frac{2}{3}x+2x^2+2x^3-3x^4\right)dx=\left[-\frac{x}{3}-\frac{x^2}{3}+\frac{2}{3}x^3+\frac{x^4}{2}-\frac{3}{5}x^5\right]_{-1}^{1}$$

$$=-\frac{1}{3}-\frac{1}{3}+\frac{2}{3}+\frac{1}{2}-\frac{3}{5}-\left(\frac{1}{3}-\frac{1}{3}-\frac{2}{3}+\frac{1}{2}+\frac{3}{5}\right)\quad\rightarrow\quad\boxed{\left(h\middle|\overline{w_3}\right)=-\frac{8}{15}}$$

$$\left(\overline{w_3}\middle|\overline{w_3}\right)=\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2dx=\int_{-1}^{1}\left(x^4-\frac{2}{3}x^2+\frac{1}{9}\right)dx=\left[\frac{x^5}{5}-\frac{2}{9}x^3+\frac{x}{9}\right]_{-1}^{1}=$$

$$=\frac{1}{5}-\frac{2}{9}+\frac{1}{9}-\left(-\frac{1}{5}+\frac{2}{9}-\frac{1}{9}\right)\quad\rightarrow\quad\boxed{\left(\overline{w_3}\middle|\overline{w_3}\right)=\frac{8}{45}}$$

$$\alpha_3=\frac{-\frac{\not 8}{15}}{\frac{\not 8}{45}}=-\frac{45}{15}\quad\rightarrow\quad\therefore\ \boxed{\alpha_3=-3}$$

- Por tanto:

$$\boxed{(h)_{B_{OG}}=\begin{bmatrix}0\\2\\-3\end{bmatrix}}\ \leftarrow\ \text{Vector de coordenadas}$$

Problema 2: Sea $P_{\leq 2}$ el espacio vectorial de los polinomios de grado menor o igual a dos con coeficientes reales, y el producto interno en $P_{\leq 2}$, definido por:

$$\left(p\middle|q\right) = a_0 b_0 + 2a_1 b_1 + 3a_2 b_2 \qquad \forall \begin{matrix} p(x) = a_0 + a_1 x + a_2 x^2 \\ q(x) = b_0 + b_1 x + b_2 x^2 \end{matrix} \in P_{\leq 2}$$

Obtener una base ortogonal de $P_{\leq 2}$, a partir de la base $B = \left\{1 + x + x^2, 1 + x, 1\right\}$.

SOLUCIÓN:

- Utilizando el proceso de Gram-Schmidt:

$$\overline{w_1} = \overline{v_1}$$

$$\therefore \boxed{\overline{w_1} = 1 + x + x^2}$$

$$\overline{w_2} = \overline{v_2} - \frac{\left(\overline{v_2}\middle|\overline{w_1}\right)}{\left(\overline{w_1}\middle|\overline{w_1}\right)}\overline{w_1}$$

$$\left(\overline{v_2}\middle|\overline{w_1}\right) = \left(1 + x\middle|1 + x + x^2\right) = 1(1) + 2(1)(1) + 3(0)(1) = \boxed{3}$$

$$\left(\overline{w_1}\middle|\overline{w_1}\right) = \left(1 + x + x^2\middle|1 + x + x^2\right) = 1(1) + 2(1)(1) + 3(1)(1) = \boxed{6}$$

$$\therefore \overline{w_2} = 1 + x - \frac{3}{6}(1 + x + x^2) = 1 - \frac{1}{2} + x - \frac{1}{2}x - \frac{1}{2}x^2 \rightarrow \boxed{\overline{w_2} = \frac{1}{2} + \frac{1}{2}x - \frac{1}{2}x^2}$$

$$\overline{w_3} = \overline{v_3} - \frac{\left(\overline{v_3}\middle|\overline{w_1}\right)}{\left(\overline{w_1}\middle|\overline{w_1}\right)}\overline{w_1} - \frac{\left(\overline{v_3}\middle|\overline{w_2}\right)}{\left(\overline{w_2}\middle|\overline{w_2}\right)}\overline{w_2}$$

$$\left(\overline{v_3}\middle|\overline{w_1}\right) = \left(1\middle|1 + x + x^2\right) = 1(1) + 2(0)(1) + 3(0)(1) = \boxed{1}$$

$$\left(\overline{v_3}\middle|\overline{w_2}\right) = \left(1\middle|\frac{1}{2} + \frac{1}{2}x - \frac{1}{2}x^2\right) = \boxed{\frac{1}{2}}$$

$$\left(\overline{w_2}\middle|\overline{w_2}\right) = \left(\frac{1}{2} + \frac{1}{2}x - \frac{1}{2}x^2\middle|\frac{1}{2} + \frac{1}{2}x - \frac{1}{2}x^2\right) = \boxed{\frac{3}{2}}$$

$$\therefore \overline{w_3} = 1 - \frac{1}{6}(1 + x + x^2) - \frac{1/2}{3/2}\left(\frac{1}{2} + \frac{1}{2}x - \frac{1}{2}x^2\right) \rightarrow \boxed{\overline{w_3} = \frac{2}{3} - \frac{1}{3}x}$$

- Finalmente: $\boxed{B_{OG} = \left\{1 + x + x^2, \frac{1}{2} + \frac{1}{2}x - \frac{1}{2}x^2, \frac{2}{3} - \frac{1}{3}x\right\}}$ ←Base ortogonal

Problema 3: Sea $P_{\leq 2}$ el espacio vectorial real de los polinomios de grado menor o igual a dos con coeficientes reales, y sea el conjunto $B=\{1, 1+x, 1+x+x^2\}$ una base de $P_{\leq 2}$. Determinar a partir de B una base ortonormal de dicho espacio, considerando el producto interno en $P_{\leq 2}$ definido por:

$$(p|q)=p(x_1)q(x_1)+p(x_2)q(x_2)+p(x_3)q(x_3) \qquad \forall \begin{array}{l} p(x)=a_1+b_1x+c_1x^2 \\ q(x)=a_2+b_2x+c_2x^2 \end{array} \in P_{\leq 2}$$

donde $x_1=-1$; $x_2=0$; $x_3=1$.

SOLUCIÓN:

- El producto interno dado es: $\boxed{(p|q)=p(-1)q(-1)+p(0)q(0)+p(1)q(1)}$

- ¿B es ortogonal?:

$$(\overline{v_1}|\overline{v_2})=[1|1+x]=(1)(0)+(1)(1)+(1)(2)=\boxed{3}\neq 0 \;\leftarrow\; B \text{ no es ortogonal}$$

- Utilizando el proceso de Gram-Schmidt:

$$\overline{w_1}=\overline{v_1}$$
$$\therefore\ \boxed{\overline{w_1}=1}$$

$$\overline{w_2}=\overline{v_2}-\frac{(\overline{v_2}|\overline{w_1})}{(\overline{w_1}|\overline{w_1})}\overline{w_1}$$

$$(\overline{v_2}|\overline{w_1})=\boxed{3}$$

$$(\overline{w_1}|\overline{w_1})=(1|1)=(1)(1)+(1)(1)+(1)(1)\rightarrow(\overline{w_1}|\overline{w_1})=\boxed{3}$$

$$\therefore\ \overline{w_2}=1+x-\frac{\cancel{3}}{\cancel{3}}(1)=\cancel{1}+x-\cancel{1} \quad\rightarrow\quad \boxed{\overline{w_2}=x}$$

$$\overline{w_3} = \overline{v_3} - \frac{\left(\overline{v_3} \middle| \overline{w_1}\right)}{\left(\overline{w_1} \middle| \overline{w_1}\right)}\overline{w_1} - \frac{\left(\overline{v_3} \middle| \overline{w_2}\right)}{\left(\overline{w_2} \middle| \overline{w_2}\right)}\overline{w_2}$$

$$\left(\overline{v_3} \middle| \overline{w_1}\right) = \left(1 + x + x^2 \middle| 1\right) = 1(1) + (1)(1) + (3)(1) = \boxed{5}$$

$$\left(\overline{v_3} \middle| \overline{w_2}\right) = \left(1 + x + x^2 \middle| x\right) = (1)(-1) + (1)(0) + (3)(1) = \boxed{2}$$

$$\left(\overline{w_2} \middle| \overline{w_2}\right) = \left(x \middle| x\right) = (-1)(-1) + (0)(0) + (1)(1) = \boxed{2}$$

$$\therefore\ \overline{w_3} = 1 + x + x^2 - \frac{5}{3}(1) - \frac{2}{2}(x) \quad \rightarrow \quad \boxed{\overline{w_3} = x^2 - \frac{2}{3}}$$

- Por tanto: $\boxed{B_{OG} = \left\{1, x, x^2 - \frac{2}{3}\right\}}$ ←Base ortogonal

- Para la base ortonormal:

$$\overline{e_1} = \frac{1}{\left\|\overline{w_1}\right\|}\overline{w_1} \quad \rightarrow \quad \therefore \overline{e_1} = \frac{1}{\sqrt{3}}(1) = \boxed{\frac{1}{\sqrt{3}}}$$

$$\overline{e_2} = \frac{1}{\left\|\overline{w_2}\right\|}\overline{w_2} \quad \rightarrow \quad \therefore \overline{e_2} = \frac{1}{\sqrt{2}}(x) = \boxed{\frac{1}{\sqrt{2}}x}$$

$$\overline{e_3} = \frac{1}{\left\|\overline{w_3}\right\|}\overline{w_3}; \quad \left(\overline{w_3} \middle| \overline{w_3}\right) = \left(x^2 - \frac{2}{3} \middle| x^2 - \frac{2}{3}\right) = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{6}{9} = \boxed{\frac{2}{3}}$$

$$\therefore\ \overline{e_3} = \frac{1}{\sqrt{\frac{2}{3}}}\left(x^2 - \frac{2}{3}\right) = \sqrt{\frac{3}{2}}\left(x^2 - \frac{2}{3}\right) = \boxed{\sqrt{\frac{3}{2}}x^2 - \sqrt{\frac{2}{3}}}$$

- Finalmente:

$$\boxed{B_{ON} = \left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}x, \sqrt{\frac{3}{2}}x^2 - \sqrt{\frac{2}{3}}\right\}} \leftarrow \text{Base ortonormal}$$